

Numerical Approximations for the Gaussian Q -Function by Sums of Exponentials

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Abstract

The accurate prediction of wireless systems' performance is a key factor in the timely adoption of new technologies and systems' design. In many cases, when evaluating the performance measures of a communication system with additive white Gaussian noise, integrals involving the Gaussian Q -function appear and closed-form solutions cannot be expressed in terms of elementary functions. This has motivated researchers to propose approximations and bounds for the Gaussian Q -function to facilitate expression manipulations. This paper gives a brief overview about the existing approximations of the Q -function. In addition, it summarizes and compares the different quadrature numerical integration techniques that can be applied in approximating the Gaussian Q -function in a tractable form as a weighted sum of exponentials.

1 Introduction

Often, when analyzing wireless systems' performance in the presence of Gaussian noise in terms of average error probabilities, the tail probability of a standard normal random variable having unit variance and zero mean occurs and is defined as

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{1}{2}t^2\right) dt = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \exp\left(-\frac{1}{2\sin^2\theta}x^2\right) d\theta \quad [\text{for } x \geq 0]. \quad (1)$$

The latter form is proposed by Craig as an integral of an exponential on a finite interval.

Several approximations and bounds for the Gaussian Q -function have been derived by researchers to facilitate performance analysis of various communication systems. They can be classified into three classes as Mills' ratio-based form (MRBF), polynomial-based form (PLBF), and sum of exponential function-based form (SEFB).

Some examples of MRBF approximations and bounds are presented in [1–5]. In [1], the authors propose accurate approximations and bounds with relatively high complexity, which makes them more suitable for improving calculation efficiency rather than to ease algebraic manipulation in analysing communication systems' performance. More accurate approximations that provide sufficient accuracy for the whole positive x -axis for $Q(x)$ and its integer powers are derived in [2,3]. Based on [2], an upper bound is later developed in [4], and a simpler approximation is obtained in [5] using Taylor series expansion.

The importance of PLBF approximations is best seen in the performance analysis of lognormal channels. The approximation in [6] is an example of polynomial approximations. A number of SEFB approximations and bounds are proposed in [7–11]. In particular, [7] proposes an approximation in the form of finite sum of exponentials and based on a semi-infinite Gauss–Hermite quadrature rule. A second-order exponential approximation is derived in [8]. In [9], the authors use the trapezoidal integration rule with optimizing the integral of relative error to yield a two-term exponential approximation. The authors in [10] uses Prony approximation to present a new exponential approximation. In [11], the composite trapezoidal rule is applied with choosing the optimal number of sub-intervals.

2 Approximations From Numerical Integration

In addition to the aforementioned approximations of the Q -function, other approximations in the form of weighted sum of exponential functions are derived herein using the various numerical integration rules that can be applied on Craig's formula. They are categorized herein as Newton–Cotes and Gaussian quadrature formulas. In general, any integral of the form $\int_u^v W(\theta) f(\theta) d\theta$, where $W(\theta)$ is some weighting function and $[u, v]$ is the domain of integration, can be approximated by a finite sum of the form

$$\int_u^v W(\theta) f(\theta) d\theta \approx \sum_{h=1}^H w_h f(\theta_h), \quad (2)$$

where $\{\theta_h\}_{h=1}^H$ are the nodes and $\{w_h\}_{h=1}^H$ are the weights of the H -point quadrature. The weights and nodes of the various numerical techniques can be found in [12].

2.1 Newton–Cotes Numerical Integration

There are two types of Newton–Cotes formulas, namely closed and open. The former uses the function value at all points, and the latter does not use them at the endpoints.

We approximate the Q -function in (1), which has the domain of integration $[0, \pi/2]$, as $\tilde{Q}(x)$ with exactly N nonzero-exponential terms, by using the Newton–Cotes rule stated in (2), at equally-spaced points. Therefore,

$$\tilde{Q}(x) = \sum_{n=1}^N \frac{1}{\pi} w_n \exp\left(-\frac{x^2}{2 \sin^2 \theta_n}\right) = \sum_{n=1}^N a_n \exp(-b_n x^2), \quad (3)$$

The numerical coefficients of the exponential summation are

$$\{(a_n, b_n)\}_{n=1}^N = \left\{ \left(\frac{1}{\pi} w_n, \frac{1}{2 \sin^2 \theta_n} \right) \right\}_{n=1}^N. \quad (4)$$

It should be noted that for the closed Newton–Cotes, when substituting the left endpoint of the integration domain in (2), the first terms become zero. Hence, in order to have exactly N nonzero-exponential terms, the number of nodes, H , must be equal to $H = N + 1$. Whereas, for the open type, $H = N$.

2.2 Gaussian Quadrature Numerical Integration

The domain of integration of (2) for the Gaussian quadrature family is $[-1, 1]$, and since the Q -function is defined over $[0, \pi/2]$ when using Craig's formula, then a change of variables is required. Therefore, the Q -function is approximated as $\tilde{Q}(x)$ with exactly N nonzero-exponential terms as follows

$$\tilde{Q}(x) = \frac{1}{4} \sum_{n=1}^N w_n \exp\left(\frac{-x^2}{2 \sin^2\left(\frac{\pi}{4} \theta_n + \frac{\pi}{4}\right)}\right) = \sum_{n=1}^N a_n \exp(-b_n x^2), \quad (5)$$

resulting in the following coefficients

$$\{(a_n, b_n)\}_{n=1}^N = \left\{ \left(\frac{1}{4} w_n, \frac{1}{2 \sin^2\left(\frac{\pi}{4} \theta_n + \frac{\pi}{4}\right)} \right) \right\}_{n=1}^N. \quad (6)$$

Five different Gaussian rules can be implemented to evaluate the Q -function numerically, namely Legendre, Chebyshev first and second kind, Radau's, and Lobatto's rule [12]. For Radau's and Lobatto's rule, the left endpoint of the integration interval is a node which evaluates the first term in (2) to zero. Therefore, for these two techniques, $H = N + 1$. Whereas, for the remaining Gaussian techniques, $H = N$.

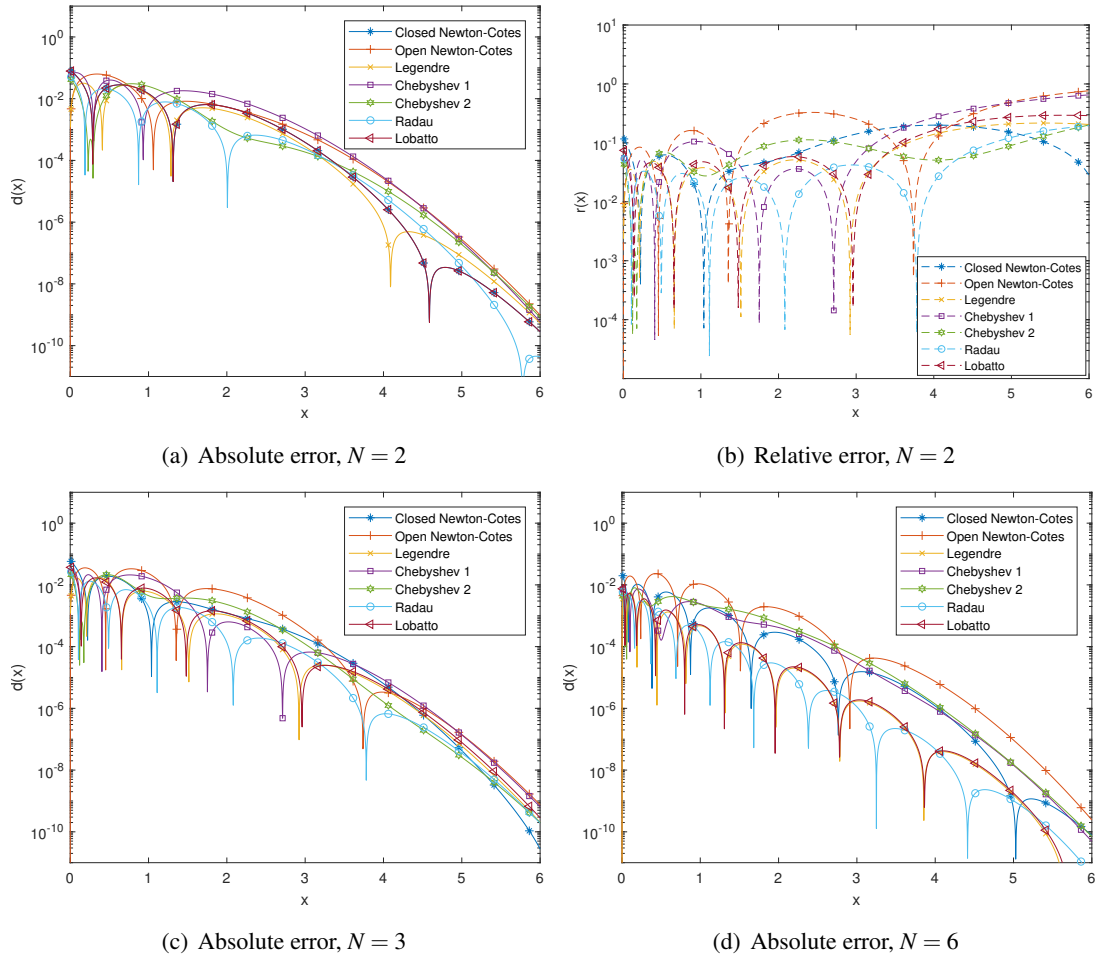


Fig. 1: comparison between the different Q -function's approximations obtained from the two types of Newton–Cotes (open and closed) and the five Gaussian quadrature rules (Legendre, Chebyshev first and second kind, Radau's, and Lobatto's rule).

3 Discussion and Conclusion

The aforementioned numerical techniques are applied to (1) and exponential approximations for the Q -function are obtained. A comparison among these techniques is done in terms of absolute and relative error in order to illustrate the accuracy achieved. They are defined respectively as

$$d(x) \triangleq \tilde{Q}(x) - Q(x) \quad \text{and} \quad r(x) \triangleq \frac{d(x)}{Q(x)} = \frac{\tilde{Q}(x)}{Q(x)} - 1. \quad (7)$$

Figs. 1(a) and 1(b) illustrate a comparison between the different numerical approximations of the Q -function with two exponential terms in terms of the absolute and relative error, respectively. It can be noted that all the presented numerical techniques have a comparable accuracy. However, some techniques outperform the others for different x -ranges, for example, for the absolute error, Legendre quadrature rule has the least error for $3.2 \leq x \leq 4.3$, whereas Chebyshev type one and the closed Newton–Cotes are more accurate for $4.3 \leq x \leq 5$. In general, as x increases and tends to infinity, the absolute error decreases and converges to zero and the absolute value of the relative error converges to 1. Moreover, we made a comparison among all Q -function's numerical approximations in terms of absolute error for $N = 2$, $N = 3$, and $N = 6$ in Figs. 1(a), 1(c) and 1(d), respectively. The accuracy of the numerical technique increases by increasing the number

of exponential terms. In general, we can see that Radau's rule has the best accuracy over a wide range of argument, and its accuracy increases considerably by increasing number of terms.

In summary, we have shown that numerical integration can be used to derive tractable approximation for the Gaussian Q -function up to the desired precision by controlling the number of exponential terms in the summation, which in turn can enable the analytical calculation of average probabilities in many applications.

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